Joint Motion

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Content

Introduction

- This presentation addresses the kinematic equations for *n*-body systems whose joints can be any of the following types
  - ❖ 1 dof translational ⟺ joint type(JT)=1
  - ❖ 3 dof translational ⟺ joint type(JT)=2
  - ❖ 1 dof rotational ⟺ joint type(JT)=3
  - ❖ 3 dof rotational ⟺ joint type(JT)=4

- MBS with Mixed joint types appear often in robotics, and complex vibration isolation mechanisms.

- The generalized coordinates chosen for the considered system is the set:

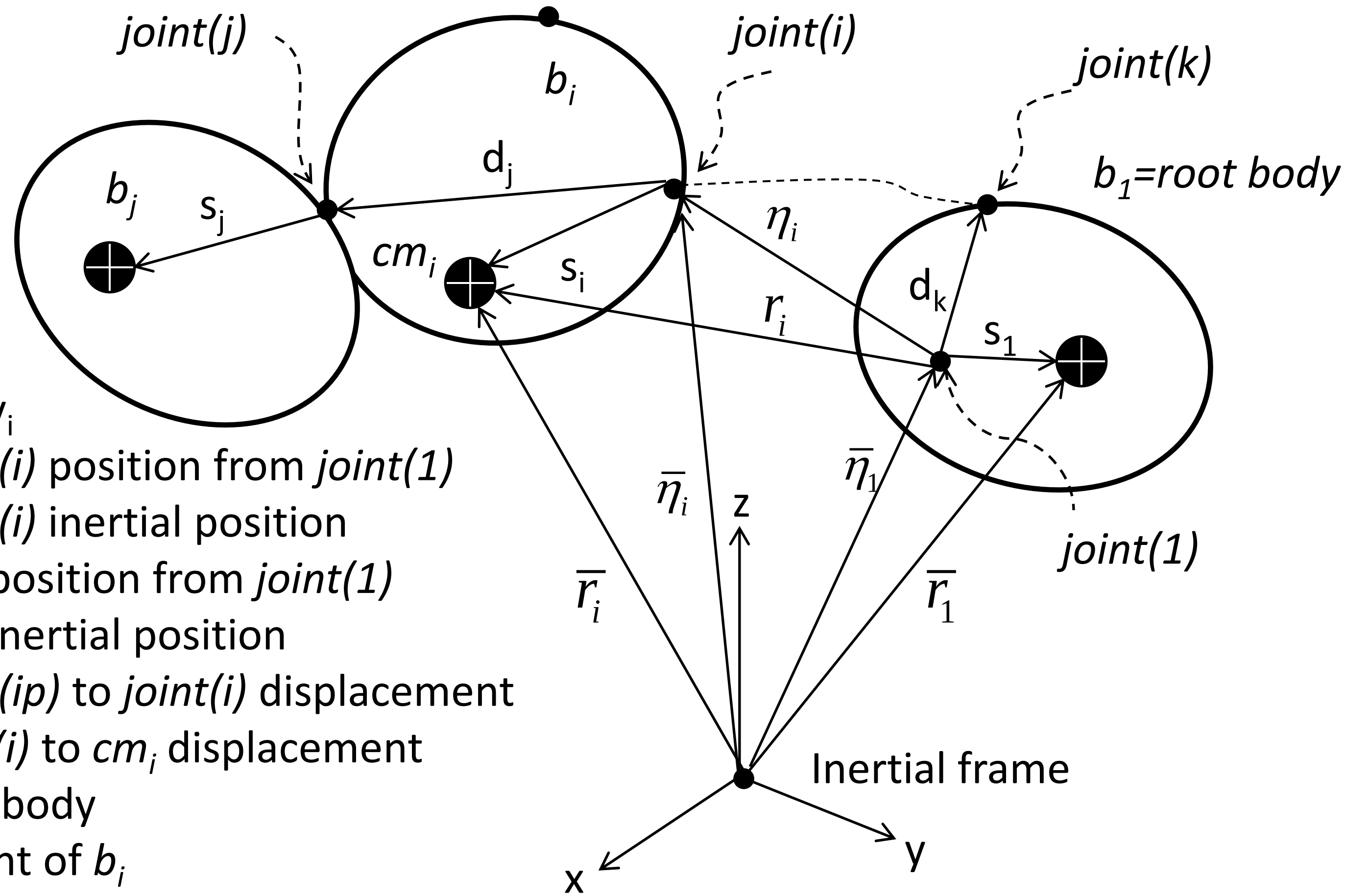
$$q = \{\gamma_i\}_{i=1:N}$$

where

$$\gamma_i = \begin{cases} \delta_i, \text{ scalar displacement;} & \text{JT}(i) = 1 \\ \mathbf{x}_i^i, \text{ } 3\times 1 \text{ displacement;} & \text{JT}(i) = 2 \\ \theta_i, \text{ joint angle;} & \text{JT}(i) = 3 \\ \tilde{\varepsilon}_i, \text{ relative attitude quaternion ;} & \text{JT}(i) = 4 \end{cases}$$

$$= \text{inboard joint coordinate of } b_i$$

Fig. 1 Joint 1 Centric Notation



b_i body$_i$
η_i *joint(i)* position from *joint(1)*
$\bar{\eta}_i$ *joint(i)* inertial position
r_i cm_i position from *joint(1)*
$\bar{r}_i$ cm_i inertial position
d_j *joint(ip)* to *joint(i)* displacement
s_j *joint(i)* to cm_i displacement
b_1 root body
b_{ip} parent of b_i

- The root body b_1 is the reference body whose position and attitude serve as the starting value to compute the same for other bodies in the system in a hierarchical manner. Generally, the choice of b_1 is arbitrary. In the case of a humanoid robot, b_1 could be the head, the torso or the hip.

- Body indexing rule used here is the Parent-First order meaning that the index of a body is always a lower integer number than the indices of its children.

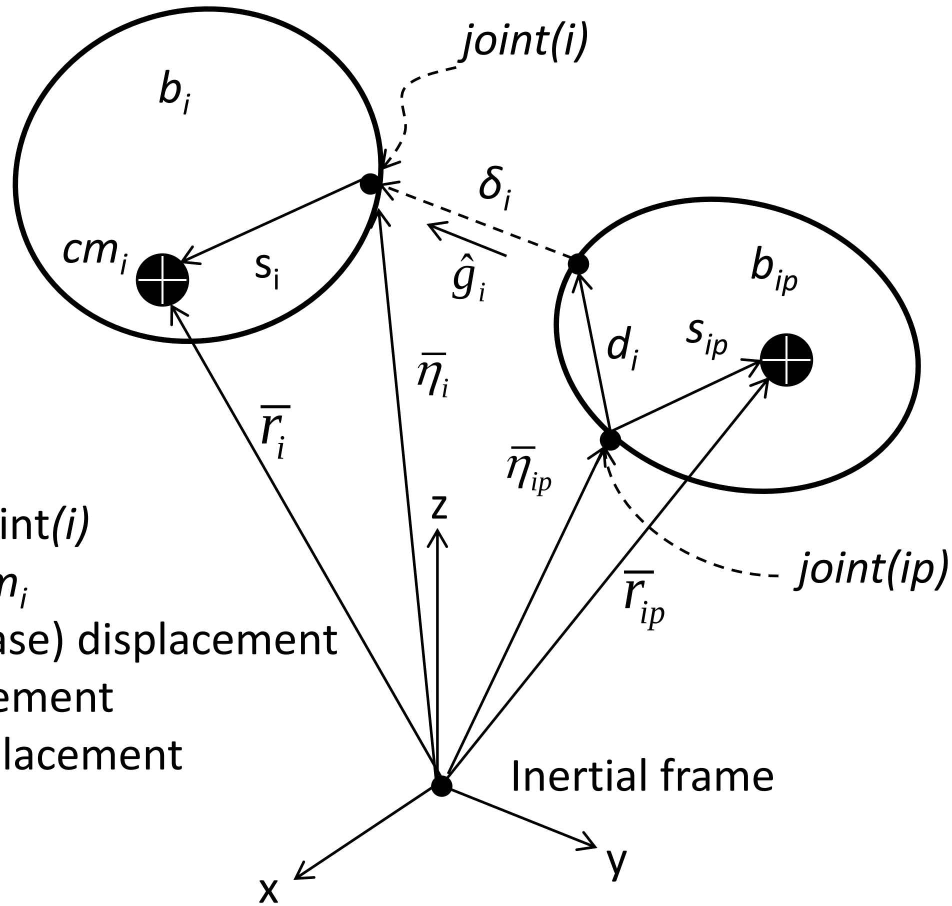
- The chain of bodies between b_1 and b_j shall be denoted as $\{i \mid i \le j\}$ or just $i \le j$.
  The less-than-or-equal relation over body indices is a topological order and not a numerical order.

- The set of bodies branching from b_j shall be denoted as $\{i \mid i \ge j\}$ or just $i \ge j$.
  The greater-than-or-equal relation over body indices is a topological order and not a numerical order.

- All vectors in the following discussion are given in the format x_j^i . The subsript j denotes the body that x belongs to and the superscript i denotes the coordinate frame that the vector is in.

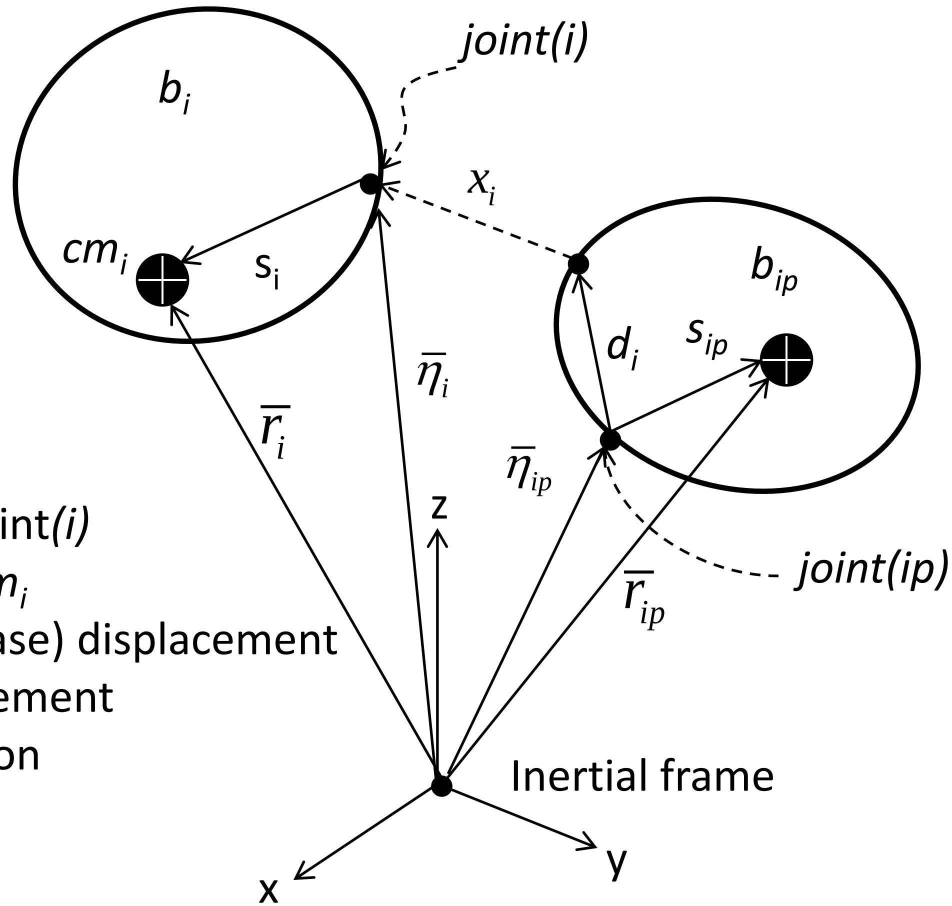
- Vectors with no superscript are given in inertial coordinates unless defined otherwise

Fig. 2 Prismatic Joint, JT=1



b_i $body_i$
b_{ip} parent of b_i
$\overline{\eta}_i$ inertial position of joint*(i)*
$\overline{r}_i$ inertial position of cm_i
d_i *joint(ip)* to *joint(i)* (base) displacement
s_i *joint(i)* to cm_i displacement
δ_i prismatic *joint(i)* displacement
$\hat{g}_i$ translational axis

Fig. 3 3 Dof Translational Joint, JT=2



b_i body$_i$

b_{ip} parent of b_i

$\bar{\eta}_i$ inertial position of joint*(i)*

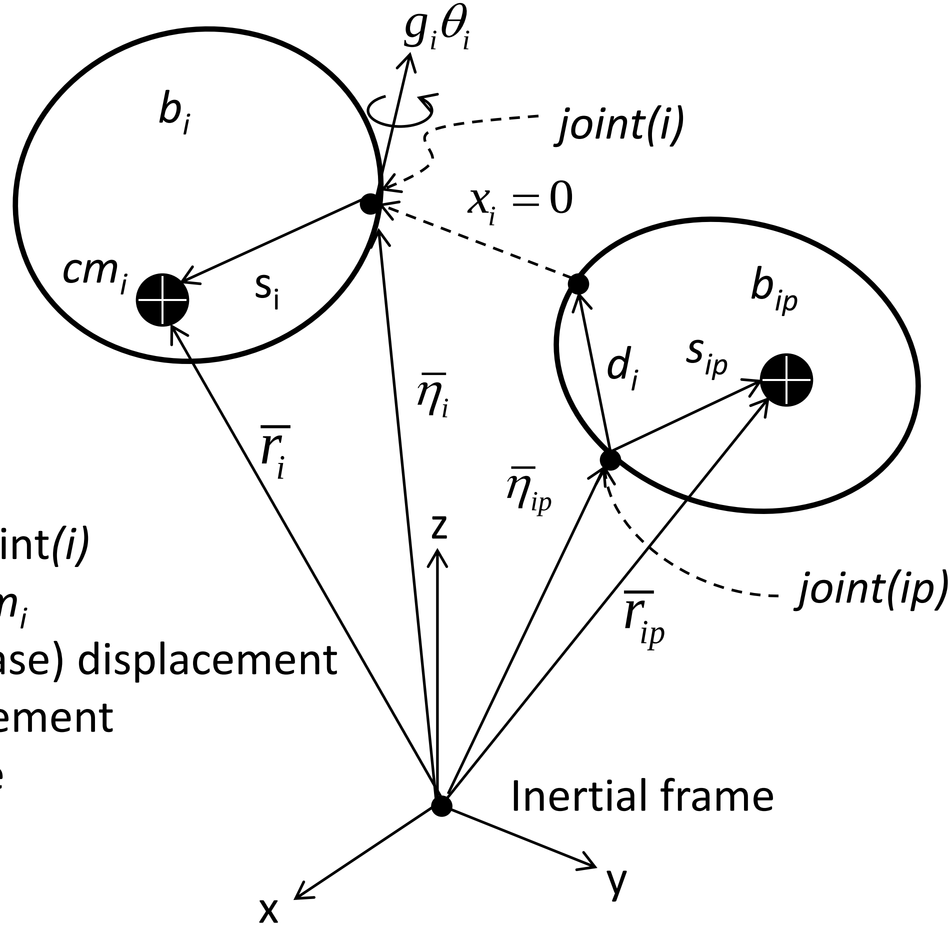
$\bar{r}_i$ inertial position of cm_i

d_i *joint(ip)* to *joint(i)* (base) displacement

s_i *joint(i)* to cm_i displacement

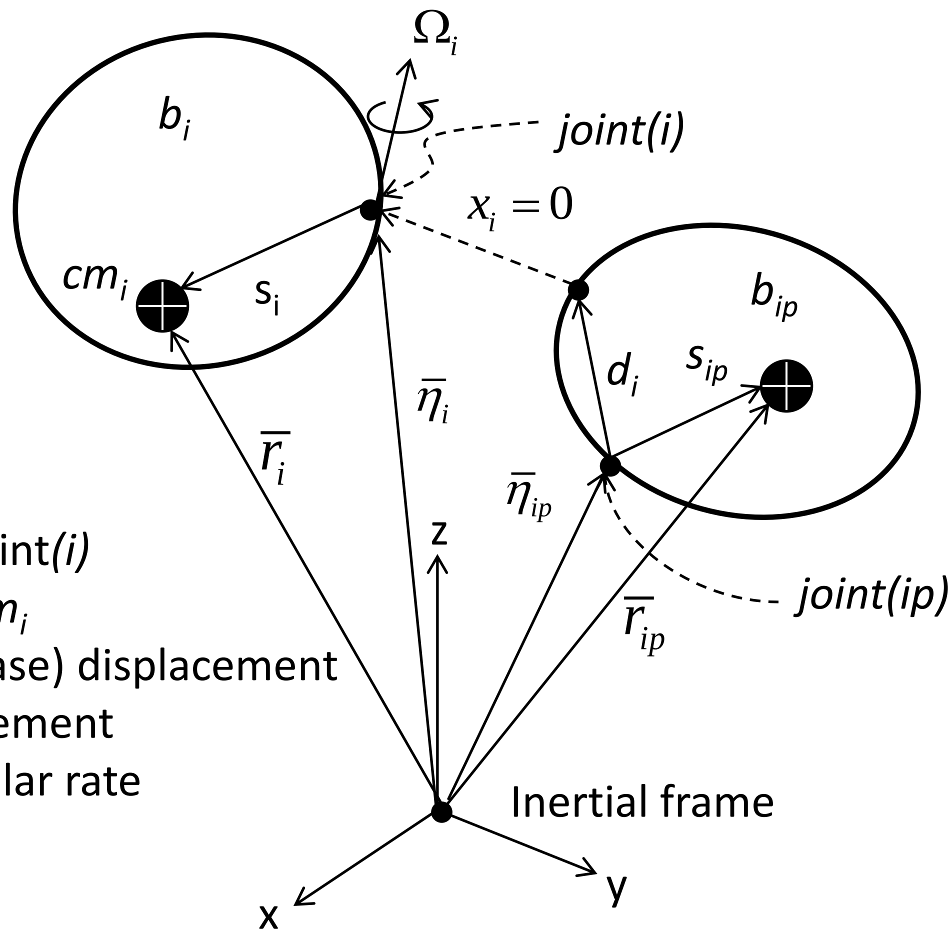
x_i 3 dof relative tranlation

Fig. 4 1 Dof Rotatioal Joint, JT=3



b_i body$_i$
b_{ip} parent of b_i
$\overline{\eta}_i$ inertial position of joint*(i)*
$\overline{r}_i$ inertial position of cm_i
d_i *joint(ip)* to *joint(i)* (base) displacement
s_i *joint(i)* to cm_i displacement
θ_i *joint(i)* rotation angle
g_i rotational axis

Fig. 5 3 Dof Rotatioal Joint, JT=4



b_i body$_i$

b_{ip} parent of b_i

$\bar{\eta}_i$ inertial position of joint*(i)*

$\bar{r}_i$ inertial position of cm_i

d_i *joint(ip)* to *joint(i)* (base) displacement

s_i *joint(i)* to cm_i displacement

Ω_i 3 dof rotational angular rate

Coordinate Frames

- A general reference frame is a 3 dimensional space with its origin defined at some point of interest. The translational motion of objects are defined by the [x, y, z] coordinates of the objects over time in that frame. The attitude motion of each body is defined by a direction cosine matrix that maps vectors fixed on that body to the reference frame over time.

- A local reference frame (*LRF*) shall mean a body fixed reference frame whose origin is located at the inboard joint of the body.

- An inertial frame is any frame in which Newtonian dynamics hold.

- Workspace frame is an inertial reference frame chosen to define the motion of a mechanism or an object such as a robot in that frame.

Attitude Equations

- A local reference frame is a body fixed three dimensional coordinate frame with its origin at the inboard joint of the body. For b_1, the LRF_1 origin is defined at an arbitrary reference point on it, called joint(1) here. See Fig. 1.

- Given $q = \{\gamma_i\}_{i=1:N}$ the direction cosine matrix C_i from b_i to the inertial frame is defined recursively by

$$C_i = C_{ip} C_i^{ip}(\gamma_i) \quad \text{for } i \geq 2 \qquad (2)$$

where $C_i^{ip}(\gamma_i) =$ relative dcm from b_i to b_{ip} frame

$ip =$ index of parent body of b_i

$C_1 = C_1^0$, 0 superscript means the inertial frame

- If joint(i) is translational (JT(i)=1 and 2)

$$C_i = C_{ip} \text{ and } C_i^{ip}(\gamma_i) = e_{3\times 3} \qquad (2a)$$

- If joint(i) is 1 dof rotational with $\gamma_i = \theta_i$ (JT(i)=3), then

$$C_i^{ip}(\theta_i) = \bar{C}_i^{ip}\left[(1-\cos(\theta_i))g_i^i g_i^{iT} + \cos(\theta_i)e + \sin(\theta_i)\tilde{g}_i^i\right] \qquad (2b)$$

where $\bar{C}_i^{ip} = C_i^{ip}(\theta_i = 0)$

$g_i^i =$ joint(i) rotation axis in LRF_i coordinates

$\tilde{g}_i^i =$ skew symmetric matrix of vector g_i^i

$e = 3\times 3$ identity matrix

- If joint(i) is 3 dof rotational with the relative quaternion $\gamma_i = \tilde{\varepsilon}_i = (a\ b\ c\ d)^T$, (JT($i$)=4) then

$$C_i^{ip}(\tilde{\varepsilon}_i) = \begin{bmatrix} a^2-b^2-c^2+d^2 & 2(ab-cd) & 2(ac+bd) \\ 2(ab+cd) & -a^2+b^2-c^2+d^2 & 2(bc-ad) \\ 2(ac-bd) & 2(bc+ad) & -a^2-b^2+c^2+d^2 \end{bmatrix} \qquad (2c)$$

- Given $q = \{\gamma_i\}_{i=1:N}$, all $\{C_i(\gamma_i)\}_{i=1:n}$ can be computed by Eqs. (2,2a,2b,2c).

Position Equations

- Given $\{C_i(\gamma_i)\}_{i=1:n}$, all body fixed vectors in the system can be expressed in inertial coordinates
- All joint and cm positions $\{\bar{\eta}_i, \bar{r}_i\}_{i=1:n}$ in the inertial reference frame per Figs. 1 and 2 can be computed as

$$\bar{\eta}_i = \begin{cases} \bar{\eta}_{ip} + d_i + \hat{g}_i \delta_i & \text{, if } \gamma_i = \delta_i \\ \bar{\eta}_{ip} + d_i + C_{ip} \mathrm{x}_i^i & \text{, if } \gamma_i = \mathrm{x}_i^i \\ \bar{\eta}_{ip} + d_i & \text{, if } \gamma_i = \theta_i \text{ or } \bar{q}_i \end{cases} \qquad (3)$$

$$\text{for } i = 2:N \text{ with } \bar{\eta}_{1p} = 0$$

$$\bar{r}_i = \bar{\eta}_i + s_i \quad \text{, for } i = 1:N \qquad (4)$$

where $d_i =$ nominal $\bar{\eta}_i$ position from $\bar{\eta}_{ip}$ when $\mathrm{x}_i^i = 0$ or $\delta_i = 0$;

$\hat{g}_i =$ slider axis; $\delta_i =$ linear displacement

$s_i =$ cm_i position from $\bar{\eta}_i$; $\mathrm{x}_i^i =$ displacement vector in b_{ip} coordinates

$\bar{\eta}_i =$ inertial position of joint(i)

$\bar{r}_i =$ inertial position of cm_i

Velocity Equations

- The system rates state $\dot{q}^* = \left\{ \dot{\gamma}_i^* \right\}_{i=1:N}$ is either obtained by solving the equations of motion or prescribed, with

$$\dot{\gamma}_i^* = \begin{cases} \dot{\delta}_i & \text{if } \gamma_i = \delta_i (\Rightarrow \Omega_i^i = 0) \text{ ; JT=1} \\ \dot{\mathrm{x}}_i^i & \text{if } \gamma_i = \mathrm{x}_i^i (\Rightarrow \Omega_i^i = 0) \text{ ; JT=2} \\ \dot{\theta}_i & \text{if } \gamma_i = \theta_i \ (\Rightarrow \dot{\mathrm{x}}_i^i = 0) \text{ ; JT=3} \\ \Omega_i^i & \text{if } \gamma_i = \overline{q}_i \ (\Rightarrow \dot{\mathrm{x}}_i^i = 0) \text{ ; JT=4} \end{cases}$$

$$\Omega_i^i = \text{ angular rate of } LRF_i \text{ w.r.t } LRF_{ip} \text{ in } LRF_i \text{ coordinates}$$

- $\dot{\tilde{\varepsilon}}_i$ relates to Ω_i^i for JT=4 as

$$\dot{\tilde{\varepsilon}}_i = \tfrac{1}{2} E_i \Omega_i^i \tag{5}$$

where $\tilde{\varepsilon}_i = [a, b, c, d]^T$ and $E_i = \begin{bmatrix} d & -c & b \\ c & d & -a \\ -b & a & d \\ -a & -b & -c \end{bmatrix}$

- Given $\dot{q}^* = \left\{ \dot{\gamma}_i^* \right\}_{i=1:N}$ all body angular velocities can be computed by

$$\omega_i = \begin{cases} \omega_{ip} & \text{JT}(i)\text{=1 or 2} \\ \omega_{ip} + g_i \dot{\theta}_i & \text{JT}(i)\text{=3} \\ \omega_{ip} + \Omega_i & \text{JT}(i)\text{=4} \end{cases} \quad (6)$$

$$\text{for } i = 2:N, \text{ with } \omega_{1p}\text{=0}$$

where $g_i =$ inboard 1 dof joint rotational axis of b_j

$$\Omega_i = C_i \Omega_i^i \text{ ; relative angular rate in inertial coordinates} \quad (7)$$

- Given $\{\dot{\bar{\eta}}_1, \omega_i\}_{i=1:n}$ the inertial velocities of all joints can be computed per Figs. 1 and 2 as

$$\dot{\bar{\eta}}_i = \begin{cases} \dot{\bar{\eta}}_{ip} + \omega_{ip} \times \bar{d}_i + \hat{\mathrm{g}}_i \dot{\delta}_i & \mathrm{JT}(i)=1 \\ \dot{\bar{\eta}}_{ip} + \omega_{ip} \times \bar{d}_i + C_{ip} \dot{\mathrm{x}}_i^i & \mathrm{JT}(i)=2 \\ \dot{\bar{\eta}}_{ip} + \omega_{ip} \times \bar{d}_i & \mathrm{JT}(i)=3 \text{ or } 4 \end{cases} \quad (8)$$

$$\text{for } i = 2:N \text{ with } \dot{\bar{\eta}}_{1p} = 0$$

where

$$\bar{d}_i = \begin{cases} C_{ip}(d_i^i + \hat{g}_i^i \delta_i) & \mathrm{JT}(i)=1 \\ C_{ip}(d_i^i + x_i^i) & \mathrm{JT}(i)=2 \\ C_{ip} d_i^i & \mathrm{JT}(i)=3 \text{ or } 4 \end{cases}$$

- Given $\{\dot{\bar{\eta}}_i, \omega_i\}_{i=1:n}$ the inertial velocities of all body *cm*'s can be computed follows

$$\dot{\bar{r}}_i = \dot{\bar{\eta}}_i + \omega_i \times s_i \quad , \text{for } i = 1:N \qquad (9)$$

Acceleration Equations

- Given $\{\dot{\omega}_1, \ddot{\gamma}_i^*\}_{i=1:N}$ all angular accelerations can be computed recursively as

$$\dot{\omega}_i = \begin{cases} \dot{\omega}_{ip} & \text{JT}(i)\text{=1 or 2} \\ \dot{\omega}_{ip} + g_i\ddot{\theta}_i + \omega_{ip} \times g_i\dot{\theta}_i & \text{JT}(i)\text{=3} \\ \dot{\omega}_{ip} + C_i\dot{\Omega}_i^i + \omega_{ip} \times C_i\Omega_i^i & \text{JT}(i)\text{=4} \end{cases} \qquad (10)$$

$$\text{for } \; i = 2:N$$

- Given $\{\ddot{\bar{\eta}}_1, \dot{\omega}_i\}_{i=1:N}$ and Eqs. (7, 8 ,9) all joint accelerations can be computed recursively as

$$\ddot{\bar{\eta}}_i = \begin{cases} \ddot{\bar{\eta}}_{ip} + \dot{\omega}_{ip} \times \bar{d}_i + \hat{g}_i\ddot{\delta}_i + \omega_{ip} \times (\omega_{ip} \times \bar{d}_i) + 2\omega_{ip} \times \hat{g}_i\dot{\delta}_i & \Leftrightarrow (\text{JT}(i)\text{=1}) \\ \ddot{\bar{\eta}}_{ip} + \dot{\omega}_{ip} \times \bar{d}_i + C_i\ddot{\text{x}}_i^i + \omega_{ip} \times (\omega_{ip} \times \bar{d}_i) + 2\omega_{ip} \times \dot{\text{x}}_i & \Leftrightarrow (\text{JT}(i)\text{=2}) \\ \ddot{\bar{\eta}}_{ip} + \dot{\omega}_{ip} \times \bar{d}_i + \omega_{ip} \times (\omega_{ip} \times \bar{d}_i) & \Leftrightarrow (\text{JT}(i)\text{=3 or 4}) \end{cases} \qquad (11)$$

$$\text{for } i = 2:n$$

- Given $\{\dot{\bar{\eta}}_i, \omega_i\}_{i=1:n}$, the cm inertial accelerations can be computed as

$$\ddot{\bar{r}}_i = \ddot{\bar{\eta}}_i + \dot{\omega}_i \times s_i + \omega_i \times (\omega_i \times s_i) \quad \text{for } i = 1:n \tag{12}$$

Other Points of Interest

- Motion of position markers on bodies may be needed for performance or constraint evaluations.

- Calculating the k-th position marker on b_j for some j :

$$p_j(k) = C_j p_j^j(k) \quad (13)$$

$$\bar{p}_j(k) = \bar{\eta}_j + p_j(k) \quad (14)$$

$$\dot{\bar{p}}_j(k) = \dot{\bar{\eta}}_j + \omega_j \times p_j(k) \quad (15)$$

$$\ddot{\bar{p}}_j(k) = \ddot{\bar{\eta}}_j + \dot{\omega}_j \times p_j(k) + \omega_j \times (\omega_j \times p_j(k)) \quad (16)$$

where $p_j^j(k) =$ position of marker(k) from joint(j) in b_j coordinates

$p_j(k) =$ inertial coordinates of $p_j^j(k)$

$\bar{p}_j(k) =$ inertial position of position marker(k)

$\dot{\bar{p}}_j(k) =$ inertial velocity of position marker(k)

$\ddot{\bar{p}}_j(k) =$ inertial acceleration of position marker(k)

- Motion of directional markers on member bodies may also be needed for performance or in constraint evaluations.

- Caculating the k-th directional marker on b_j for some j:

$$u_j(k) = C_j u_j^j(k) \tag{17}$$

$$\dot{u}_j(k) = \omega_j \times u_j(k) \tag{18}$$

$$\ddot{u}_j(k) = \dot{\omega}_j \times u_j(k) + \omega_j \times (\omega_j \times u_j(k)) \tag{19}$$

where $u_j^j(k) =$ unit vector of direction marker(k) on b_j in b_j coordinate

$u_j(k) =$ inertial coordinates of $u_j^j(k)$

$\dot{u}_j(k) =$ inertial velocity (turning rate) of $u_j(k)$

$\ddot{u}_j(k) =$ inertial acceleration of $u_j(k)$

Examples

- A ball joint is one with the coordinate $\gamma_j = \bar{q}_j$.
- A slider joint is one with the coordinate $\gamma_j = \delta_j \in \mathrm{R}$
- A universal joint consists of two serially linked joints of b_{jp} and b_j such that

$$(\gamma_{jp}, \gamma_j) = (\theta_{jp}, \theta_j) \text{ with } g_j^T g_{jp} = 0 \text{ and } d_j = 0_{3\times1}$$

- A 6 dof joint consists of two serial joints of b_{jp} and b_j such that

$$(\gamma_{jp}, \gamma_j) = (\mathrm{x}_{jp}^{jp}, \bar{q}_j), \text{ with } d_j = 0$$

Summary

- The kinematics equations in attitude, position, velocity and acceleration have been presented for a rigid joint-connected multibody system that has different types of joints. These joints can have either 1 or 3 dofs and are either rotational or translational. The kinematic equations are defined recursively and conditionally based on the type of joint.

- Kinematics equations for the position and directional markers have been presented for the purpose of performance and constraint evaluations.