Generalized Mass Matrix of MBS

Concurrent Dynamics International

September 2017

Content

Introduction

- Generalized mass matrix has been an important component in solving the dynamics equations of an *N*-body mechanism up until 1990's due to the prevailing order(N^3) algorithms. With the rise of order(*N*) recursive methods of solving the dynamics equations after 2000's, the need for that matrix became somewhat less demanding from a dynamics solving point of view.
- System mass matrix in a control system design is often assumed given or known. The latter is generally false. In those cases, it is important to know how to derive it correctly and efficiently.
- The factored form of the system mass matrix shown here makes it possible to derive its inverse in a factored form. See Refs. [1,2]. This factored inverse mass matrix in turn makes it possible to solve the equations of motion in order(*N*) manner.

- We shall consider the system mass matrix of a rigid multibody system that has only single axis rotational motion between joint connected bodies. The inboard joint of the root body is connected to the ground. A large group of mechanisms is covered with this joint description that includes robots, mechanical arms, cars, cranes, gimbaled antennas, suspension and transmission systems

- Body indexing rule used here is the Parent-First order meaning that the index of a body is always a lower integer number than the indices of its children.

- We choose the set of generalized coordinates and system rates for this multibody system as

$$q = \{\theta_i\}_{i=1:N}$$

$$\dot{q} = \{\dot{\theta}_i\}_{i=1:N}$$

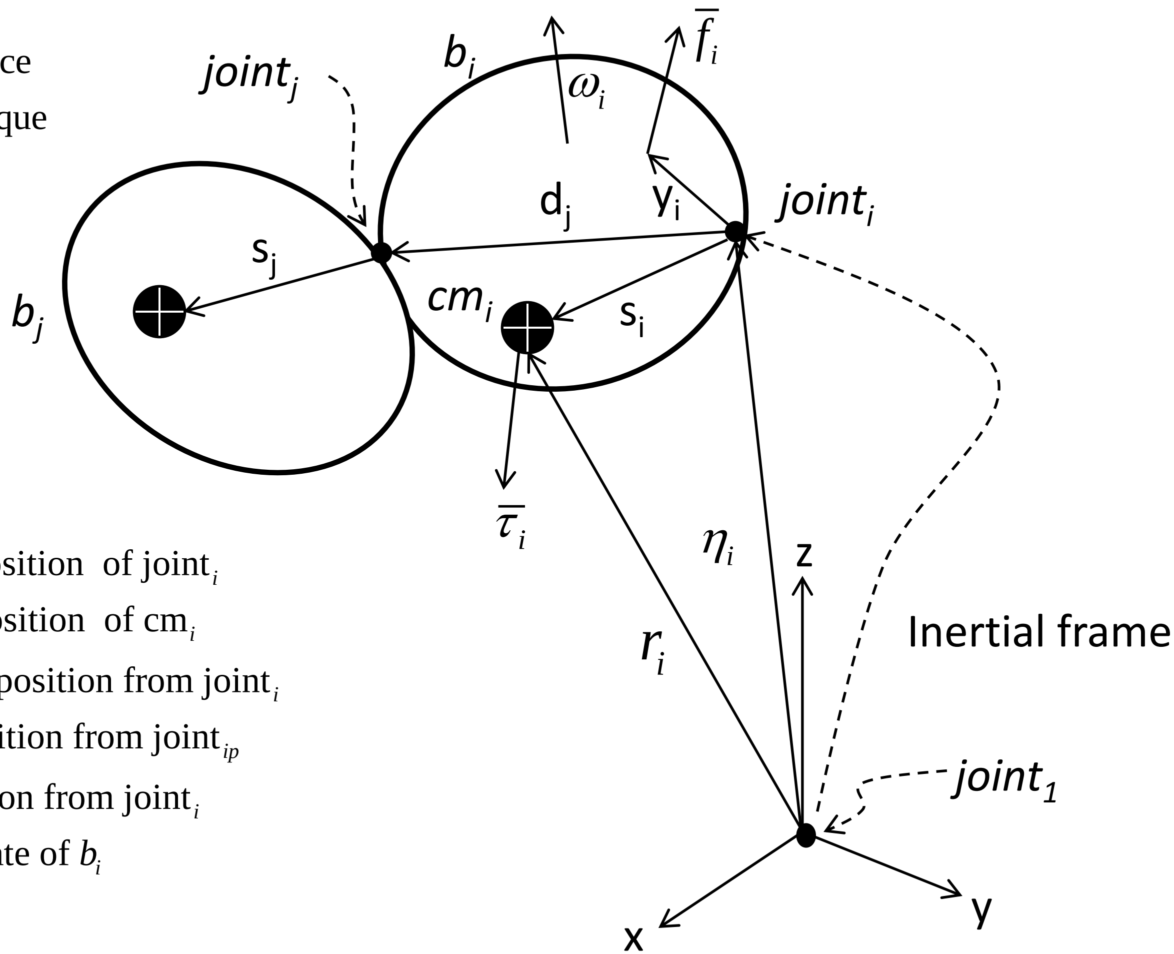
where $\theta_i =$ inboard joint angle of b_i

- From here, we will cover 1) the kinematic equations and the Jacobian, 2) incidence matrix operator, 3) energy equation and 4) generalized mass matrix.

Fig. 1 Notations

$\overline{f}_i$ external force

$\overline{\tau}_i$ external torque



b_i body i

η_i inertial position of joint_i

r_i inertial position of cm_i

y_i $\overline{f}_i$ impact position from joint_i

d_i joint_i position from joint_{ip}

s_i cm_i position from joint_i

ω_i angular rate of b_i

- The root body b_1 is the reference body whose position and attitude serve as the starting value to compute the same for other bodies in the system in a hierarchical manner. In the following, b_1 is connected to the ground

- The chain of bodies between b_1 and b_j shall be denoted as $\{i \mid i \leq j\}$ or just $i \leq j$. The less-than-or-equal relation over body indices is a topological order and not a numerical order.

- The set of bodies branching from b_j shall be denoted as $\{i \mid i \geq j\}$ or just $i \geq j$. The greater-than-or-equal relation over body indices is a topological order and not a numerical order.

- All vectors in the following discussion are given in the format x_j^i . The subsript j denotes the body that x belongs to and the superscript i denotes the coordinate frame that the vector is in.

- Vectors with no superscript are given in inertial coordinates unless defined otherwise

- A stacked vector y is a column vector whose elements are also vectors. The latter can be of different sizes. We define this stacked vector as

$$y = [y_1, y_2, y_3, \cdots, y_N], y_i \text{ is a vector}$$

where $\text{size}(y) = len(y) \times 1, \ len(y) = \sum_{i=1}^{N} len(y_i)$

For example:

Let $y_1 = v_x \ , y_2 = \begin{bmatrix} s_x \\ s_y \end{bmatrix}, y_3 = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$

Then $y = [y_1, y_2, y_3] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} v_x \\ s_x \\ s_y \\ w_x \\ w_y \\ w_z \end{bmatrix}$

- Skew matrix notation: $\tilde{a} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$, if $a = \left[a_x, a_y, a_z \right]$

- $e = k \times k$ identity matrix, size k depends on context

- The relations $\{ >, \; <, \; \geq, \; \leq \}$ shall mean topological order in the following expressions when used to group bodies, unless stated otherwise.

- $\overline{i} \;=\; \{\alpha \mid parent(\alpha) = i\}$, indices of children of b_i

Influence Matrices $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}^T$

- Influence matrices , $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}^T$, are defined by the parent child relations of the considered mechanism and a block diagonal matrix $\mathcal{A}$:

$$[\Phi_{\mathcal{A}}]_{i,j} = \begin{cases} \mathcal{A}_i \text{ if } & b_j = \text{parent of } b_i \\ 0_{6\times 6} & \text{otherwise} \end{cases} \qquad \text{Def. 1}$$

$$[\Phi_{\mathcal{A}}^T]_{i,j} = \begin{cases} \mathcal{A}_j^T \text{ if } & b_i = \text{parent of } b_j \\ 0_{6\times 6} & \text{otherwise} \end{cases} \qquad \text{Def. 2}$$

where $\mathcal{A} = \text{diag}\{\mathcal{A}_i\}_{i=1:N}$, a block diagonal matrix, with $\mathcal{A}_i \in \mathrm{R}^{k\times k}$, $\mathcal{A}_1 = 0$

$k =$ integer > 1

- Forward influence matrix, $\Phi_{\mathcal{A}}$ is strictly lower triangular.
- Backward influence matrix, $\Phi_{\mathcal{A}}^T$ is strictly upper triangular.

- Each body in a tree-configured mechanism has one parent body except for b_1. Thus, each row of $\Phi_{\mathcal{A}}$ has only one non-zero submatrix entry and the first row is all zeros.

- Given that $\Phi_{\mathcal{A}}$ is square and strictly triangular, it is nilpotent of degree *m*, i.e.

$$\Phi_{\mathcal{A}}^{m} = 0$$

where, $m =$ length of the longest link in the system from b_1, and $m \leq N$.

- Let $\mathcal{A} = diag\{\mathcal{A}_i\}_{i=1:N}, \mathcal{A}_i \in R^{k\times k}, \mathcal{A}_1$=0, and $x = col[x_i],\ z = col[z_i],$ with $x_i, z_i \in R^k, k > 1.$

Prop1. $z = \Phi_{\mathcal{A}} x$ is a column vector representation of the forward element-by-element (parent-to-child) calculations

$$z_i = \mathcal{A}_i x_{ip} \text{ for } i = 1:N \qquad (1)$$

Prop2. $z = \Phi_{\mathcal{A}}^{T} x$ is a column vector representation of the backward element-by-element (child-to-parent) calculations

$$z_i = \sum_{j \in \bar{i}} \mathcal{A}_j^{T} x_j \text{ for } i = 1:N \qquad (2)$$

- If $\Phi_{\mathcal{A}}$ is square and strictly lower triangular, then $\Psi_{\mathcal{A}} = (e - \Phi_{\mathcal{A}})^{-1}$ exists and can be expressed as

$$\Psi_{\mathcal{A}} = e + \Phi_{\mathcal{A}} + \Phi_{\mathcal{A}}^{2} + \cdots + \Phi_{\mathcal{A}}^{m-1}$$

where $m =$ nilpotency of $\Phi_{\mathcal{A}}$

- It follows that $\Phi_{\mathcal{A}}^{T}$ is square and strictly upper triangular, and $\Psi_{\mathcal{A}}^{T} = (e - \Phi_{\mathcal{A}}^{T})^{-1}$ exists and can be expressed as

$$\Psi_{\mathcal{A}}^{T} = e + \Phi_{\mathcal{A}}^{T} + \Phi_{\mathcal{A}}^{T,2} + \cdots + \Phi_{\mathcal{A}}^{T,m-1}$$

- The power series expansion of $\Psi_{\mathcal{A}}$ and $\Psi_{\mathcal{A}}^{T}$ serve to show the existence of $(e + \Phi_{\mathcal{A}})^{-1}$ and $(e + \Phi_{\mathcal{A}}^{T})^{-1}$. More efficient way of computing $\Psi_{\mathcal{A}} x$ and $\Psi_{\mathcal{A}}^{T} x$ for any $x \in \mathrm{R}^{6N}$ is shown shortly.

O(N) Recursive Operators

- The operator $\Psi_{\mathcal{A}}$ is a function of the given system parent-child relations embedded in $\Phi_{\mathcal{A}}$.

- Computing expressions involving $\left(\Psi_{\mathcal{A}} x\right)$ or $\left(\Psi^{T}_{\mathcal{A}} x\right)$ by direct matrix-vector multiplication is not efficient.

- The next two procedures to compute the two operations are order(N).

Algorithm for $z = \Psi_{\mathcal{A}} x$

- A stacked vector given by $z = \Psi_{\mathcal{A}} x$ is the result of solving $z = \Phi_{\mathcal{A}} z + x$ for z, where $\Phi_{\mathcal{A}}$ is square and strictly lower triangular per Def 1. Thus, given x, the elements of z are obtained by the algorithm below :

    1. set $z_1 = x_1$ since the first row of $\Phi_{\mathcal{A}}$ is zero
    2. for $i = 2 : N$

       $z_i = \mathcal{A}_i z_{ip} + x_i \quad ;$

       end

- $\Psi_{\mathcal{A}}$ is an order(N) operator, since $z = \Psi_{\mathcal{A}} x$ is computed in one 'N do-loop'.

Algorithm for $z = \Psi^T_{\mathcal{A}} x$

- A stacked vector given by $z = \Psi^T_{\mathcal{A}} x$ is the result of solving $z = \Phi^T_{\mathcal{A}} z + x$ for z, where $\Phi^T_{\mathcal{A}}$ is square and strictly upper triangular. Thus, given x, the elements of z are obtained by the algorithm below :

    1. set $z = x$ since the first column of $\Phi^T_{\mathcal{A}}$ is zero
    2. for $i = N : 2$

        $z_{ip} := z_{ip} + \mathcal{A}_i^T z_i$;

        end

- $\Psi^T_{\mathcal{A}}$ is an order(N) operator, since $z = \Psi^T_{\mathcal{A}} x$ is computed in one 'N do-loop'.

- In light of Def. 1 and Prop. 1, the joint velocity vectors can be written in a stacked vector form as

$$v = \Phi_D v + G\dot{q} \ \in \mathrm{R}^{6N\times 1} \quad (3)$$

where $v = [v_1, v_2, v_3, \cdots, v_N]$

$v_i = [\omega_i, \dot{\eta}_i]$, joint velocity pair for b_i, see Figure 1

$\Phi_D =$ incidence matrix given the *D matrix*

$$D = diag\{D_i\}_{i=1:N}, D_1 = 0_{6\times 6}, D_i = \begin{bmatrix} e_{3\times 3} & 0 \\ -\tilde{d}_i & e_{3\times 3} \end{bmatrix}_{i=2:N}$$

$e =$ identity matrix

$G = diag\{G_i\}_{i=1:N} \ , G_i = [g_i, 0_{3\times 1}] \in R^{6\times 1}$

$g_i \in R^{3\times 1}$, free motion axis of joint$_i$

$\dot{q} = [\dot{\theta}_1, \cdots, \dot{\theta}_N]$, rate state

- The stacked joint velocity vector in Eq. (3) can be solved as

$$v = \Psi_D G\dot{q} \tag{4}$$

where $\Psi_D = (e - \Phi_D)^{-1}, e = 6N \times 6N$ identity matrix

- The body velocities $[\bar{v}_i]_{i=1:N}$ relate to the joint velocities $[v_i]_{i=1:N}$ in a stacked vector form as

$$\bar{v} = Sv \ \in \mathrm{R}^{6N\times1} \tag{5}$$

where $\bar{v} = [\omega_i, \dot{r}_i]_{i=1:N}$, body velocity pairs

$v = [\omega_i, \dot{\eta}_i]_{i=1:N}$, joint velocity pairs

$S = diag[S_i]_{i=1:N}$

$$S_i = \begin{bmatrix} e_{3\times3} & 0 \\ -\tilde{s}_i & e_{3\times3} \end{bmatrix}$$

$s_i =$ joint$_i$ to cm$_i$ displacement vector in inertial coordinates, see Fig. 1

- Given Eqs. (4) and (5), the body rate vectors relate to the rate state, $\dot{q}$, as

$$\overline{v} = S\Psi_D G\dot{q} \qquad (6)$$

- Thus the Jacobian of the body velocities defined by $\frac{\partial \overline{v}}{\partial \dot{q}}$ is

$$J(q) = \frac{\partial \overline{v}}{\partial \dot{q}} = S\Psi_D G \in \mathrm{R}^{6N \times N} \qquad (7)$$

Kinetic Energy and Mass Matrix

- Kinetic energy of an N-body system is

$$T = \tfrac{1}{2}\sum_{i=1}^{N}\int_{b_i}(\dot{l}+\dot{r}_i)^T(\dot{l}+\dot{r}_i)dm \qquad (8)$$

where $l =$ displacement of dm from cm_i

$$\dot{l} = \omega_i \times l$$

- Equation (8) reduces to

$$T = \tfrac{1}{2}\sum_{i=1}^{N}(\omega_i^T I_i \omega_i + m_i \dot{r}_i^T \dot{r}_i) \qquad (9)$$

where $I_i = -\int_{b_i}\tilde{l}\tilde{l}dm,\ \int_{b_i} l\,dm = 0,\ m_i = \int_{b_i} dm$

- Eq. (9) reduces to

$$T = \tfrac{1}{2}\sum_{i=1}^{N} \overline{v}_i^T \overline{M}_i \overline{v}_i \qquad (10)$$

where $\overline{M}_i = \begin{bmatrix} I_i & 0 \\ 0 & m_i e \end{bmatrix}, \overline{v}_i = [\omega_i, \dot{r}_i]$

- By stacking the body velocity vectors in Eq. (10), the latter becomes

$$T = \tfrac{1}{2}\overline{v}^T \overline{M}\, \overline{v} \qquad (11)$$

where $\overline{M} = diag[\overline{M}_i]_{i=1:N} \in R^{6N\times 6N}, \overline{v} = [\overline{v}_i]_{i=1:N}$

- By Eq. (6) the kinetic energy becomes

$$T = \tfrac{1}{2}\dot{q}^T G^T \Psi_D^T S^T \bar{M}\, S \Psi_D G \dot{q}$$
$$= \tfrac{1}{2}\dot{q}^T G^T \Psi_D^T M \Psi_D G \dot{q} \qquad (12)$$

where $M = diag[M_i]_{i=1:N} \quad \in R^{6N \times 6N}$

$$M_i = S_i^T \bar{M}_i S_i = \begin{bmatrix} I_i - m_i \tilde{s}_i \tilde{s}_i & m_i \tilde{s}_i \\ -m_i \tilde{s}_i & m_i e \end{bmatrix} \in R^{6\times 6}$$

- Eq. (12) simplifies to

$$T = \tfrac{1}{2}\dot{q}^T \mathcal{M} \dot{q} \qquad (13)$$

with $\mathcal{M} = G^T \Psi_D^T M \Psi_D G \quad \in R^{N\times N}$ (14)

- The second partial derivative of T w.r.t. $\dot{q}$ per Eq. (13) yields the system mass matrix as

$$\mathcal{M} = \frac{\partial^2 T}{\partial^2 \dot{q}} \qquad (15)$$

Computing System Mass Matrix

Procedure to compute $\mathcal{M}$ per (14):

1. Compute $J = \Psi_D G$

   with $J = [J(:,1)\, J(:,2) \cdots J(:,N)]$

   $G = [G(:,1)\, G(:,2) \cdots G(:,N)]$

   for $i = 1:N$

   $J(:,i) = \Psi_D G(:,i)$; per Alg.1

   end

$\Rightarrow$

2. Compute $Q = MJ$

   with $Q = [Q(:,1)\, Q(:,2) \cdots Q(:,N)]$

   $M = diag\{M(k), k = 1:N\}$

   for $i = 1:N$

   $Q(:,i) = MJ(:,i)$;

   end

3. $\mathcal{M} = J^T Q$, end

Q1: what is the order of the above procedure?

Q2: can it be made order(N)?

- Ans1: Steps 1, 2 and 3 are each order(N^2), therefore the procedure is order(N^2)
- Ans2: No. Computation of $\mathcal{M}$ is order(N^2) per Ans1.

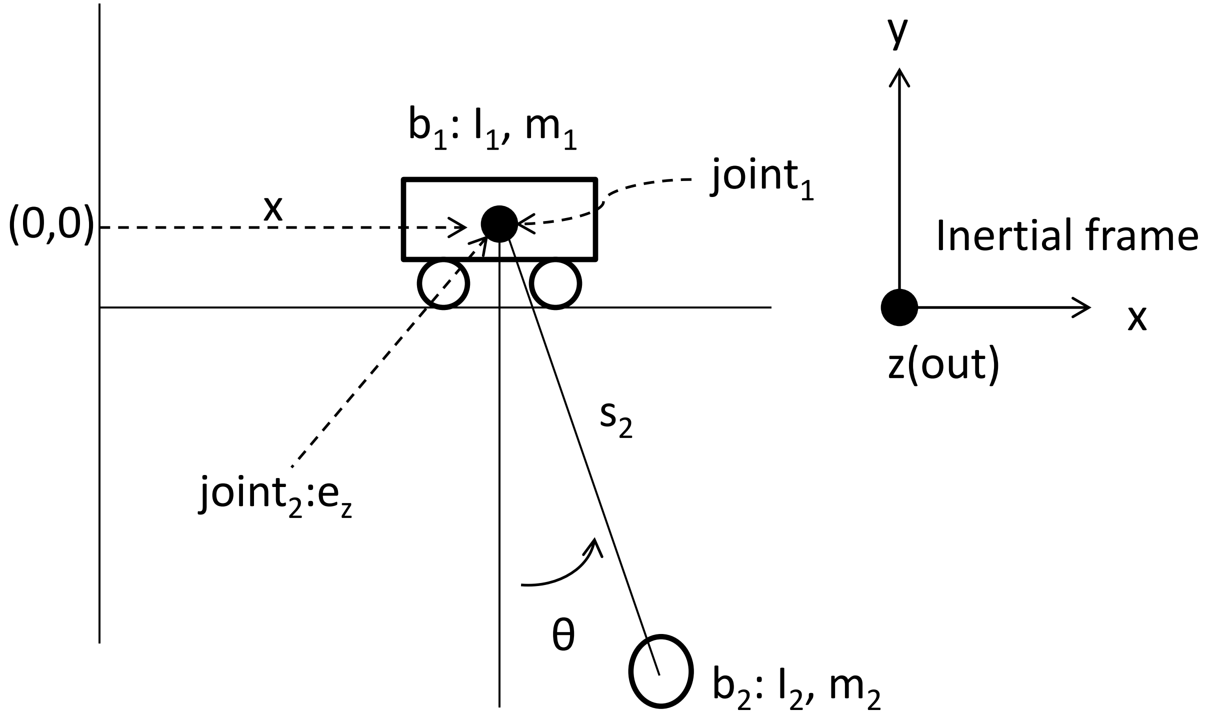
Examples

- Each system mass matrix of the four examples that follow is hand derived to illustrate the explicit expression of the elements of the matrix for the model of interest.

- For practical purposes use the given system mass matrix procedure to compute the numerical value of that matrix.

- Test the procedure-based matrix result against the values of hand derived system mass matrix to verify that they are identical.

Example 1
Pendulum with a Moving Base



Model Parameters

$D = diag\{0_{6\times6}, D_2\}$, with $D_2 = e_{6\times6}$, given $d_2 = 0$, $\omega_1 = 0, \eta_1 = xe_x$

$s_1 = 0, s_2 = L(\sin(\theta)e_x - \cos(\theta)e_y)$

$$G = diag\{G_1, G_2\} = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}_{12\times2}, \text{ with } G_1 = \begin{bmatrix} 0 \\ e_x \end{bmatrix}, G_2 = \begin{bmatrix} e_z \\ 0 \end{bmatrix} \quad \text{(M1)}$$

$$\Phi_D = \begin{bmatrix} 0 & 0 \\ D_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ e & 0 \end{bmatrix}, \ 0 = 0_{6\times6} \ \text{nilpotency}(\Phi_D) = 2 \quad \text{(M2)}$$

$$\Psi_D = (e - \Phi_D)^{-1} = e + \Phi_D = \begin{bmatrix} e_{6\times6} & 0 \\ e_{6\times6} & e_{6\times6} \end{bmatrix} \quad \text{(M3)}$$

$$M = diag\{M_1, M_2\} \quad \text{(M4)}$$

- Rate state for this example is $\dot{q} = [\dot{x}, \dot{\theta}_2]$

- Given (M1) and (M3), we have

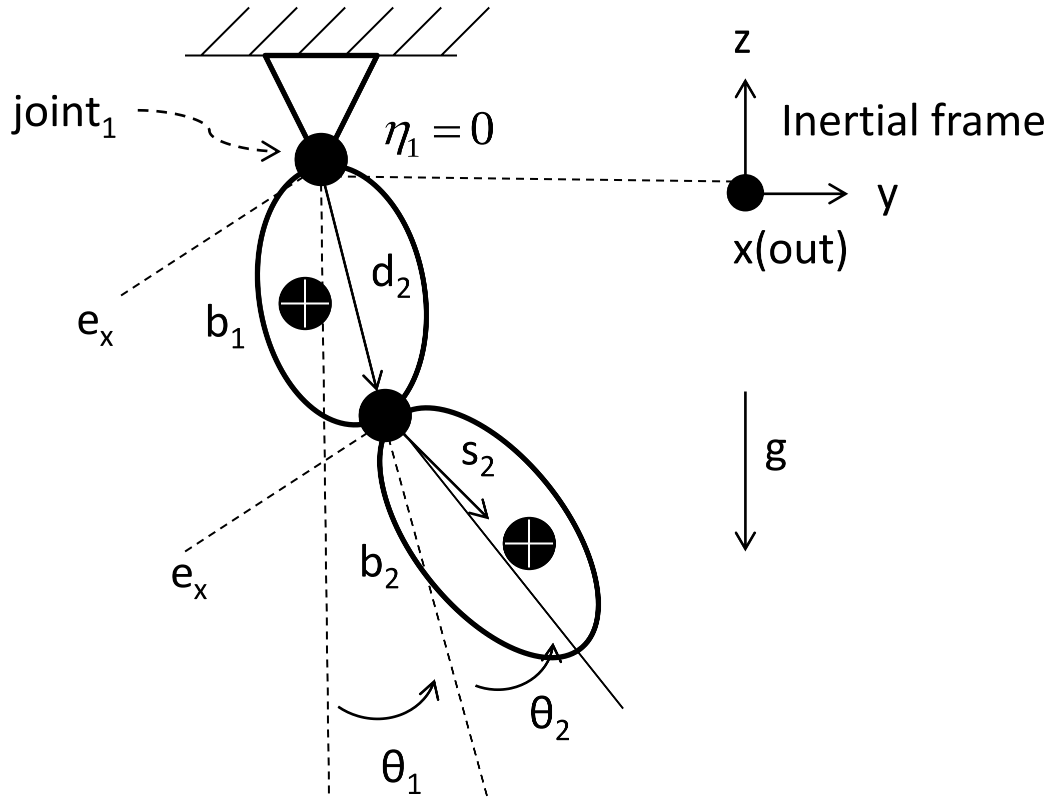
$$\Psi_D G = \begin{bmatrix} G_1 & 0 \\ G_1 & G_2 \end{bmatrix}_{12\times 2} \qquad \text{(M5)}$$

- The system mass matrix of the pendulum on a moving base per (13) is

$$\begin{aligned}
\mathcal{M} &= G^T \Psi_D^T M \Psi_D G \\
&= \begin{bmatrix} G_1^T & G_1^T \\ 0 & G_2^T \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ G_1 & G_2 \end{bmatrix} \\
&= \begin{bmatrix} G_1^T M_1 G_1 + G_1^T M_2 G_1 & G_1^T M_2 G_2 \\ G_2^T M_2 G_1 & G_2^T M_2 G_2 \end{bmatrix} \\
&= \begin{bmatrix} [0, e_x]^T M_1 [0, e_x] + [0, e_x]^T M_2 [0, e_x] & [0, e_x]^T M_2 [e_z, 0] \\ [e_z, 0]^T M_2 [0, e_x] & [e_z, 0]^T M_2 [e_z, 0] \end{bmatrix} \\
&= \begin{bmatrix} m_1 + m_2 & m_2 L \cos(\theta) \\ m_2 L \cos(\theta) & I_2^{zz} + m_2 L^2 \end{bmatrix}_{2\times 2}
\end{aligned}$$

QED

Example 2
Double Pendulum



Model Parameters

$$D = diag\{0_{6\times6}, D_2\},\ \text{with } d_1 = 0,\ D_2 = \begin{bmatrix} e & 0 \\ -\tilde{d}_2 & e \end{bmatrix},\ \text{with } \eta_1 = 0$$

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}_{12\times2},\ \text{with } G_1 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6\times1}, G_2 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6\times1} \qquad \text{(N1)}$$

$$\Phi_D = \begin{bmatrix} 0 & 0 \\ D_2 & 0 \end{bmatrix},\ \text{nilpotency}(\Phi_D) = 2 \qquad \text{(N2)}$$

$$\Psi_D = (e - \Phi_D)^{-1} = e + \Phi_D = \begin{bmatrix} e_{6\times6} & 0 \\ D_2 & e_{6\times6} \end{bmatrix} \qquad \text{(N3)}$$

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}_{12\times12} \qquad \text{(N4)}$$

- Rate state for this example is $\dot{q} = [\dot{\theta}_1, \dot{\theta}_2]$

- By (N1) and (N3) we have

$$\Psi_D G = \begin{bmatrix} G_1 & 0 \\ D_2 G_1 & G_2 \end{bmatrix}_{12\times 2} \qquad \text{(N5)}$$

- The system mass matrix of the double pendulum per (13) is

$$\begin{aligned} \mathcal{M} &= G^T \Psi_D^T M \Psi_D G \\ &= \begin{bmatrix} G_1^T & G_1^T D_2^T \\ 0 & G_2^T \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ D_2 G_1 & G_2 \end{bmatrix} \\ &= \begin{bmatrix} G_1^T M_1 G_1 + G_1^T D_2^T M_2 D_2 G_1 & G_1^T D_2^T M_2 G_2 \\ G_2^T M_2 D_2 G_1 & G_2^T M_2 G_2 \end{bmatrix} \\ &= \begin{bmatrix} [e_x, 0]^T M_1 [e_x, 0] + [e_x, -\tilde{d}_2 e_x]^T M_2 [e_x, -\tilde{d}_2 e_x] & [e_x, -\tilde{d}_2 e_x]^T M_2 [e_x, 0] \\ [e_x, 0]^T M_2 [e_x, -\tilde{d}_2 e_x] & [e_x, 0]^T M_2 [e_x, 0] \end{bmatrix}_{2\times 2} \end{aligned}$$

(N6)

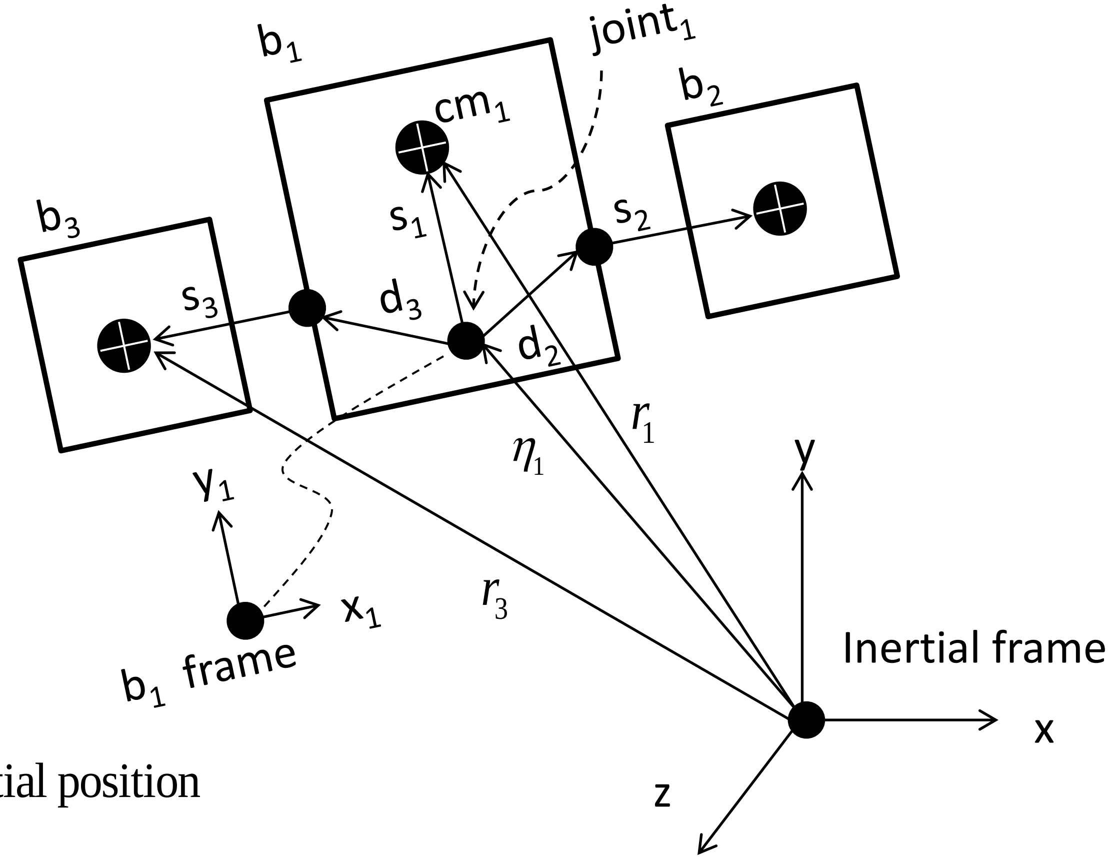
- Equation (N6) can be expanded with some algebraic manipulations to be

$$\mathcal{M} = \begin{bmatrix} e_x^T (I_1 + I_2 - m_1 \tilde{r}_1 \tilde{r}_1 - m_2 \tilde{r}_2 \tilde{r}_2) e_x & e_x^T (I_2 - m_2 \tilde{r}_2 \tilde{s}_2) e_x \\ e_x^T (I_2 - m_2 \tilde{s}_2 \tilde{r}_2) e_x & e_x^T (I_2 - m_2 \tilde{s}_2 \tilde{s}_2) e_x \end{bmatrix}$$

where $r_1 = s_1, \quad r_2 = d_2 + s_2$

QED

Example 3
A Satellite with Two Arrays



$r_i = cm_i$ inertial position

Model Parameters

$$D = diag\{0_{6\times6}, D_2, D_3\},\ \text{with } D_j = \begin{bmatrix} e_{3\times3} & 0 \\ -\tilde{d}_j & e_{3\times3} \end{bmatrix}_{j=2,3} \quad \text{(O1)}$$

$$e_x = C_1^{in}[1,0,0] \quad \text{(O2)}$$

$$G = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix}_{18\times8},\ \text{with } G_1 = e_{6\times6}, G_2 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6\times1}, G_3 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6\times1} \quad \text{(O3)}$$

$$\Phi_D = \begin{bmatrix} 0 & 0 & 0 \\ D_2 & 0 & 0 \\ D_3 & 0 & 0 \end{bmatrix}_{18\times18} \quad \text{with } 0 = 0_{6\times6},\ \text{nilpotency}(\Phi_D) = 2 \quad \text{(O4)}$$

$$\Psi_D = (e - \Phi_D)^{-1} = (e + \Phi_D) = \begin{bmatrix} e & 0 & 0 \\ D_2 & e & 0 \\ D_3 & 0 & e \end{bmatrix}_{18\times18} \quad \text{(O5)}$$

- Rate state for this satellite is $\dot{q} = [v_1, \dot{\theta}_1, \dot{\theta}_2]$ with $v_1 = [\omega_1, \dot{\eta}_1]$

- Given (O3) and (O5), we have

$$\Psi_D G = \begin{bmatrix} e & 0 & 0 \\ D_2 & G_2 & 0 \\ D_3 & 0 & G_3 \end{bmatrix}_{18\times 8} \qquad \text{(O6)}$$

- The system mass matrix per (13) is

$$\mathcal{M} = \begin{bmatrix} e & D_2^T & D_3^T \\ 0 & G_2^T & 0 \\ 0 & 0 & G_3^T \end{bmatrix} \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} e & 0 & 0 \\ D_2 & G_2 & 0 \\ D_3 & 0 & G_3 \end{bmatrix}$$

$$= \begin{bmatrix} M_1 + D_2^T M_2 D_2 + D_3^T M_3 D_3 & D_2^T M_2 G_2 & D_3^T M_3 G_3 \\ G_2^T M_2 D_2 & G_2^T M_2 G_2 & 0 \\ G_3^T M_3 D_3 & 0 & G_3^T M_3 G_3 \end{bmatrix}_{8\times 8} \qquad \text{(O7)}$$

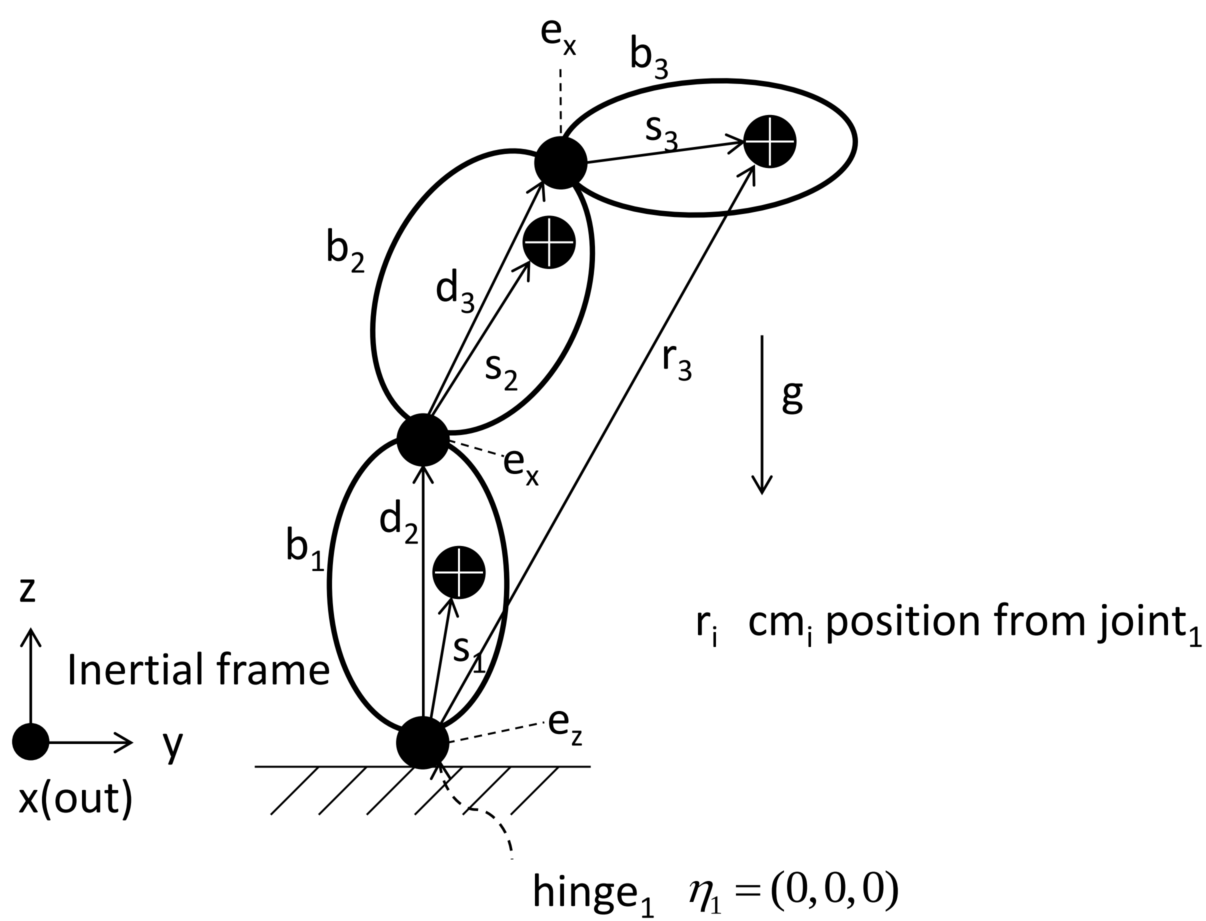
- Given (O1) to (O3), and after some algebraic manipulations, (O7) becomes

$$\mathcal{M} = \begin{bmatrix} \sum_{j=1}^{3}(I_j - m_j\tilde{r}_j\tilde{r}_j) & \bar{m}\tilde{c} & (I_2 - m_2\tilde{r}_2\tilde{s}_2)e_x & (I_3 - m_3\tilde{r}_3\tilde{s}_3)e_x \\ -\bar{m}\tilde{c} & \bar{m}e & -m_2\tilde{s}_2e_x & -m_3\tilde{s}_3e_x \\ e_x^T(I_2 - m_2\tilde{s}_2\tilde{r}_2) & e_x^T m_2\tilde{s}_2 & e_x^T(I_2 - m_2\tilde{s}_2\tilde{s}_2)e_x & 0 \\ e_x^T(I_3 - m_3\tilde{s}_3\tilde{r}_3) & e_x^T m_3\tilde{s}_3 & 0 & e_x^T(I_3 - m_3\tilde{s}_3\tilde{s}_3)e_x \end{bmatrix}$$

where $r_1 = s_1, r_j = d_j + s_j$ for $j = 2:3, \bar{m} = \sum_{j=1}^{3} m_j, c = \frac{\sum_{j=1}^{3} m_j r_j}{\bar{m}}$

QED

Example 4
3-Link Arm



Model Parameters

$$D = diag\{0_{6\times6}, D_2, D_3\},\ D_j = \begin{bmatrix} e & 0 \\ -\tilde{d}_j & e \end{bmatrix}_{j=2,3} \text{ and } \eta_1 = 0$$

$$G = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{bmatrix}_{18\times3},\ \text{with } G_1 = \begin{bmatrix} e_z \\ 0 \end{bmatrix}_{6\times1}, G_2 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6\times1}, G_3 = \begin{bmatrix} e_x \\ 0 \end{bmatrix}_{6\times1} \quad \text{(P1)}$$

$$\Phi_D = \begin{bmatrix} 0 & 0 & 0 \\ D_2 & 0 & 0 \\ 0 & D_3 & 0 \end{bmatrix}_{18\times18},\ \text{nilpotency}(\Phi_D) = 3 \quad \text{(P2)}$$

$$\Psi_D = (e - \Phi_D)^{-1} = e + \Phi_D + \Phi_D^2 = \begin{bmatrix} e_{6\times6} & 0 & 0 \\ D_2 & e_{6\times6} & 0 \\ D_3 D_2 & D_3 & e_{6\times6} \end{bmatrix} \quad \text{(P3)}$$

$$M = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix}_{18\times18} \quad \text{(P4)}$$

- Generalized rates for this arm is $\dot{q} = [\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3]$

- Given Eqs. (P1) and (P2), we have

$$\Psi_D G = \begin{bmatrix} G_1 & 0 & 0 \\ D_2 G_1 & G_2 & 0 \\ D_{3:2} G_1 & D_3 G_2 & G_2 \end{bmatrix}_{18\times 3} \qquad \text{(P5)}$$

where $D_{3:2} = D_3 D_2 = \begin{bmatrix} e & 0 \\ -(d_3 + d_2)^* & e \end{bmatrix}_{6\times 6}$

- System mass matrix of the 3-link arm is

$$\mathcal{M} = G^T \Psi_D^T M \Psi_D G$$

$$= \begin{bmatrix} G_1^T & G_1^T D_2^T & G_1^T D_{3:2}^T \\ 0 & G_2^T & G_2^T D_3^T \\ 0 & 0 & G_3^T \end{bmatrix} \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{bmatrix} \begin{bmatrix} G_1 & 0 & 0 \\ D_2 G_1 & G_2 & 0 \\ D_{3:2} G_1 & D_3 G_2 & G_3 \end{bmatrix}$$

$$= \begin{bmatrix} G_1^T M_1 G_1 + G_1^T D_2^T M_2 D_2 G_1 + G_1^T D_{3:2}^T M_3 D_{3:2} G_1 & G_1^T D_2^T M_2 G_2 + G_1^T D_{3:2}^T M_3 D_3 G_2 & G_1^T D_{3:2}^T M_3 G_3 \\ G_2^T M_2 D_2 G_1 + G_2^T D_3^T M_3 D_{3:2} G_1 & G_2^T M_2 G_2 + G_2^T D_3^T M_3 D_3 G_2 & G_2^T D_3^T M_3 G_3 \\ G_3^T M_3 D_{3:2} G_1 & G_3^T M_3 D_3 G_2 & G_3^T M_3 G_3 \end{bmatrix}_{3\times 3}$$

(P6)

- Equation (P6) can be expanded to be

$$\mathcal{M} = \begin{bmatrix} e_z^T \sum_{j=1}^{3} (I_j - m_j \tilde{r}_j \tilde{r}_j) e_z & e_z^T \sum_{j=2}^{3} (I_j - m_j \tilde{r}_j (\tilde{r}_j - \tilde{\eta}_2)) e_x & e_z^T (I_3 - m_3 \tilde{r}_3 \tilde{s}_3) e_x \\ e_x^T \sum_{j=2}^{3} (I_j - m_j (\tilde{r}_j - \tilde{\eta}_2) \tilde{r}_j) e_z & e_x^T \sum_{j=2}^{3} (I_j - m_j (\tilde{r}_j - \tilde{\eta}_2)(\tilde{r}_j - \tilde{\eta}_2)) e_x & e_x^T (I_3 - m_3 (\tilde{r}_3 - \tilde{\eta}_2) \tilde{s}_3) e_x \\ e_x^T (I_3 - m_3 \tilde{s}_3 \tilde{r}_3) e_z & e_x^T (I_3 - m_3 \tilde{s}_3 (\tilde{r}_3 - \tilde{\eta}_2)) e_x & e_x^T (I_3 - m_3 \tilde{s}_3 \tilde{s}_3) e_x \end{bmatrix}_{3\times 3} \tag{P7}$$

where $\eta_1 = 0, \quad \eta_2 = d_2, \quad \eta_3 = d_2 + d_3$

$r_1 = s_1, \quad r_2 = d_2 + s_2, \quad r_3 = d_2 + d_3 + s_3$

QED

Summary

- The system mass matrix of a multibody system is shown to be a second order partial derivative of the system kinetic energy with respect to the generalized velocities.

- This mass matrix can be expressed explicitly in a factored form based on the Jacobian matrix of the system body velocities. That expression depends on the influence matrix operator Φ_D and the associated forward and backward recursive operators (Ψ_D, Ψ_D^T) .

- The algorithm presented for computing the system mass matrix is more versatile and efficient than the case-by-case hand derivation of the same matrix for any mechanical system as shown by the examples given.

References

1. Rodriguez, G., Jain, A., Kreutz-Delgado, K., “Spatial Operator Algebra for Manipulator Modeling and Control”, The International Journal of Robotics Research, Vol. 10, No. 4, August, 1991.
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